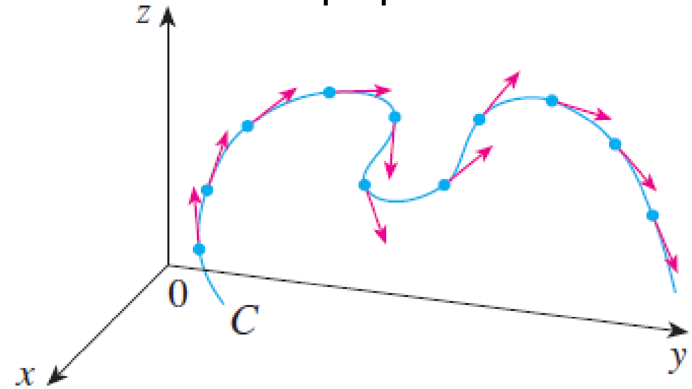
Close Tue: 13.3 (part 2)
Close Thu: 13.4, 14.1
Exam 1 will be returned Tuesday

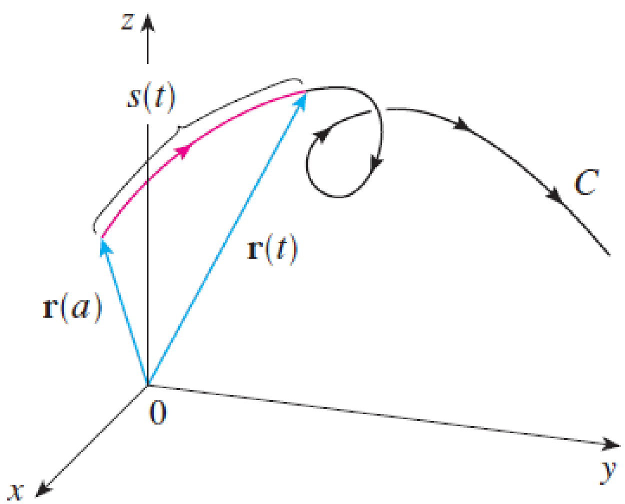
$$K = \left|\frac{dT}{ds}\right| = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \text{curvature}$$



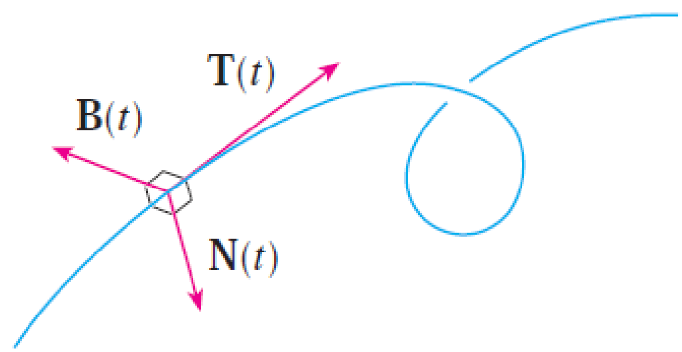
13.3 Analyzing 3D Curves

Given $\boldsymbol{r}(t) = < x(t), y(t), z(t) >$

$$s(t) = \int_0^t |\boldsymbol{r}'(t)| dt = \text{distance}$$



Today we will also find the *T*, *N* and *B* vectors below (TNB Frame)



The **curvature** at a point, *K*, is a measure of how quickly a curve is changing direction at that point. That is, we define

$$K = \frac{\text{change in direction}}{\text{change in distance}}$$

Roughly, how much does your direction change if you move a small amount ("one inch") along the curve?

$$K \approx \left|\frac{\vec{T_2} - \vec{T_1}}{\text{"one inch"}}\right| = \left|\frac{\Delta\vec{T}}{\Delta s}\right|$$

1st shortcut:

$$K(t) = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

2nd shortcut

$$K(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Example: Find the curvature function for $\boldsymbol{r}(t) = \langle t, \cos(2t), \sin(2t)\rangle$.

$$\boldsymbol{r}'(t) = \langle 1, -2\sin(2t), 2\cos(2t)\rangle$$
$$\boldsymbol{r}''(t) = \langle 0, -4\cos(2t), 4\sin(2t)\rangle$$

$$|\boldsymbol{r}'(t)| = \sqrt{1 + 4\sin^2(2t) + 4\cos^2(2t)}$$

so $|\boldsymbol{r}'(t)| = \sqrt{5}$

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \langle -8, -4\sin(2t), -4\cos(2t)\rangle$$

So $|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)| = \sqrt{64 + 16} = \sqrt{80}$

$$\frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\sqrt{80}}{\sqrt{5}^3} = \sqrt{\frac{80}{125}} = 0.8$$

This curve has constant curvature!

Proof of short cuts:

Lemma:

$\boldsymbol{T}$ and $\boldsymbol{T}'$ are always orthogonal.

Proof of lemma:

Since $\boldsymbol{T} \cdot \boldsymbol{T} = |\boldsymbol{T}|^2 = 1$, we can differentiate both sides to get

$$\boldsymbol{T}' \cdot \boldsymbol{T} + \boldsymbol{T} \cdot \boldsymbol{T}' = 0.$$

So $2\boldsymbol{T} \cdot \boldsymbol{T}' = 0$. Thus, $\boldsymbol{T} \cdot \boldsymbol{T}' = 0$. (QED)

Theorem: $\frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}$

Proof of theorem:

Since $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$, we have

$$\boldsymbol{r}'(t) = |\boldsymbol{r}'(t)|\boldsymbol{T}(t).$$

Differentiating this gives (prod. rule):

$$\boldsymbol{r}''(t) = |\boldsymbol{r}'(t)|'\boldsymbol{T}(t) + |\boldsymbol{r}'(t)|\boldsymbol{T}'(t).$$

Take cross-prod. of both sides with $\vec{\boldsymbol{T}}$:

$$\boldsymbol{T}\times\boldsymbol{r}'' = \left|\boldsymbol{r}'\right|' (\boldsymbol{T}\times\boldsymbol{T}) + \left|\boldsymbol{r}'\right| (\boldsymbol{T}\times\boldsymbol{T}').$$

Since $\boldsymbol{T}\times\boldsymbol{T} = <0,0,0>$ (why?) and $\boldsymbol{T} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|}$, we have

$$\frac{\boldsymbol{r}'\times\boldsymbol{r}''}{|\boldsymbol{r}'|} = |\boldsymbol{r}'|\,(\boldsymbol{T}\times\boldsymbol{T}').$$

Taking the magnitude gives (why?)

$$\frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|} = |\boldsymbol{r}'|\,|\boldsymbol{T}\times\boldsymbol{T}'| = |\boldsymbol{r}'|\,|\boldsymbol{T}||\boldsymbol{T}'|sin\left(\frac{\pi}{2}\right),$$

Since $|\boldsymbol{T}| = 1$, we have

$$|\boldsymbol{T}'| = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^2}$$

Therefore

$$K = \left|\frac{d\boldsymbol{T}}{ds}\right| = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3}.$$

Note: To find curvature for a 2D function, *y = f(x)*,we can form a 3D vector function as follows

$$\boldsymbol{r}(x) = \langle x, f(x), 0\rangle$$

so $\boldsymbol{r}'(x) = \langle 1, f'(x), 0\rangle$ and

$$\boldsymbol{r}''(x) = \langle 0, f''(x), 0\rangle$$

$$|\boldsymbol{r}'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \langle 0,0, f''(x)\rangle$$

Thus,

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

Example:

Consider $x = t, y = t^2, z = 0.$

At what point (x, y, z) is the curvature maximum?

The TNB-Frame:

As we proved earlier,

$\vec{\boldsymbol{T}}'(t)$ is always orthogonal to $\vec{\boldsymbol{T}}(t)$.

Not only is it orthogonal, it also points 'inwardly' relative to whichever way you are curving.

Thus, if we get the inward pointing **unit** vector by:

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \text{principal unit normal}$$

We also define

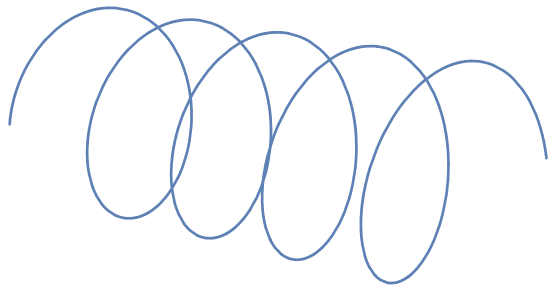
$$\vec{\boldsymbol{B}}(t) = \vec{\boldsymbol{T}}(t) \times \vec{\boldsymbol{N}}(t) = \text{binormal}$$

Some TNB Facts:

- All have length one.
- We define the **normal plane** as parallel to $\vec{\boldsymbol{N}}(t)$ and $\vec{\boldsymbol{B}}(t)$ and is orthogonal to $\vec{\boldsymbol{T}}(t)$ (and $\vec{\boldsymbol{r}}'(t)$).
- $\vec{\boldsymbol{T}}(t)$ and $\vec{\boldsymbol{N}}(t)$ point in the tangent and inward directions, respectively, so they give a good approximation of the "plane of motion". This "plane of motion" that goes through a point on the curve and is parallel to $\vec{\boldsymbol{T}}(t)$ and $\vec{\boldsymbol{N}}(t)$ is called the "**osculating plane**" ("osculating" means "kissing")
- $\vec{\boldsymbol{T}}(t)$, $\vec{\boldsymbol{N}}(t)$, $\vec{\boldsymbol{r}}'(t)$, and $\vec{\boldsymbol{r}}''(t)$ are ALL parallel to the osculating plane.
- $\vec{\boldsymbol{B}}(t)$ is orthogonal to the osculating plane, it is also orthogonal to ALL the vectors $\vec{\boldsymbol{T}}(t)$, $\vec{\boldsymbol{N}}(t)$, $\vec{\boldsymbol{r}}'(t)$, and $\vec{\boldsymbol{r}}''(t)$

Example:

$$\vec{\boldsymbol{r}}(t) = < 2\sin(3t), t, 2\cos(3t) >$$



Find

1. Find the normal plane at t = π.
2. $\vec{\boldsymbol{T}}(\pi)$
3. $\vec{\boldsymbol{N}}(\pi)$
4. $\vec{\boldsymbol{B}}(\pi)$
5. Find the osculating plane at t = π.

Summary of 3D Curve Measurement Tools:

Given $\vec{\boldsymbol{r}}(t) = < x(t), y(t), z(t) >$

$\vec{\boldsymbol{r}}'(t) =$ a tangent vector

$$s(t) = \int_0^t |\vec{\boldsymbol{r}}'(t)| dt$$

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3}$$

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \text{unit tangent}$$

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \text{principal unit normal}$$

$$\vec{\boldsymbol{B}}(t) = \vec{\boldsymbol{T}}(t) \times \vec{\boldsymbol{N}}(t) = \text{binormal}$$

Tangent Line:
Through curve in direction of a tangent vector.

Normal Plane:
Through curve orthogonal to a tangent vector.

Osculating Plane:
Through curve parallel to both $\vec{\boldsymbol{r}}'(t)$ and $\vec{\boldsymbol{r}}''(t)$